

REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH COMMUTING RESTRICTED JACOBI OPERATORS

EUNMI PAK, YOUNG JIN SUH AND CHANGHWA WOO

ABSTRACT. In this paper, we have considered a new commuting condition, that is, $(R_\xi\phi)S = S(R_\xi\phi)$ (resp. $(\bar{R}_N\phi)S = S(\bar{R}_N\phi)$) between the restricted Jacobi operator $R_\xi\phi$ (resp. $\bar{R}_N\phi$), and the Ricci tensor S for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. In terms of this condition we give a complete classification for Hopf hypersurfaces M in $G_2(\mathbb{C}^{m+2})$.

INTRODUCTION

The complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ are defined as the set of all complex two-dimensional linear subspaces in \mathbb{C}^{m+2} . It is a Hermitian symmetric space of rank 2 with compact irreducible type. Remarkably, it is equipped with both a Kähler structure J and a quaternionic Kähler structure \mathfrak{J} (not containing J) satisfying $JJ_\nu = J_\nu J$ ($\nu = 1, 2, 3$), where $\{J_\nu\}_{\nu=1,2,3}$ is an orthonormal basis of \mathfrak{J} . In this paper, we assume $m \geq 3$ (see Berndt and Suh [3] and [4]).

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ and N denote a local unit normal vector field to M . By using the Kähler structure J of $G_2(\mathbb{C}^{m+2})$, we can define a structure vector field by $\xi = -JN$, which is said to be a *Reeb vector field*. If ξ is invariant under the shape operator A , it is said to be *Hopf*. In addition, M is said to be a *Hopf hypersurface* if every integral curve of M is totally geodesic. By the formulas in [7, Section 2], it can be easily seen that ξ is Hopf if and only if M is Hopf. From the quaternionic Kähler structure \mathfrak{J} of $G_2(\mathbb{C}^{m+2})$, there naturally exist *almost contact 3-structure* vector fields defined by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Next, let us denote by $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ a 3-dimensional distribution in a tangent space $T_p M$ at $p \in M$, where \mathcal{Q} stands for the orthogonal complement of \mathcal{Q}^\perp in $T_p M$. Thus the tangent space of M at $p \in M$ consists of the direct sum of \mathcal{Q} and \mathcal{Q}^\perp , that is, $T_p M = \mathcal{Q} \oplus \mathcal{Q}^\perp$.

For two distributions $[\xi] = \text{Span}\{\xi\}$ and \mathcal{Q}^\perp , we may consider two natural invariant geometric properties under the shape operator A of M , that is, $A[\xi] \subset [\xi]$ and $A\mathcal{Q}^\perp \subset \mathcal{Q}^\perp$. By using the result of Alekseevskii [1], Berndt and Suh [3]

¹2010 *Mathematics Subject Classification*: Primary 53C40; Secondary 53C15.

²*Key words*: real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, structure Jacobi operator, normal Jacobi operator, Ricci tensor, commuting condition.

* This work was supported by Grant Proj. No. NRF-2011-220-C00002 from National Research Foundation of Korea. The first author by Grant Proj. No. NRF-2012-R1A2A2A01043023 and the third author is supported by NRF Grant funded by the Korean Government (NRF-2013-Fostering Core Leaders of Future Basic Science Program).

have classified all real hypersurfaces with these invariant properties in $G_2(\mathbb{C}^{m+2})$ as follows:

Theorem A. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and \mathcal{Q}^\perp are invariant under the shape operator of M if and only if*

- (A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
- (B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

In the case of (A) in Theorem A, we want to say M is of Type (A). Similarly, in the case of (B) in Theorem A, we say M is of Type (B).

Until now, by using Theorem A, many geometers have investigated some characterizations of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with geometric quantities like shape operator, structure (or normal) Jacobi operator, Ricci tensor, and so on. Commuting Ricci tensor means that the Ricci tensor S and the structure tensor field ϕ commute each other, that is, $S\phi = \phi S$. From such a point of view, Suh [13] has given a characterization of real hypersurfaces of Type (A) with commuting Ricci tensor

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold (\bar{M}, \bar{g}) is an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. It is defined by $(\bar{R}_X(Y))(p) = (\bar{R}(Y, X)X)(p)$, where \bar{R} denotes the curvature tensor of \bar{M} and X, Y denote any vector fields on \bar{M} . It is known to be a self-adjoint endomorphism on the tangent space $T_p\bar{M}$, $p \in \bar{M}$. Clearly, each tangent vector field X to \bar{M} provides a Jacobi operator with respect to X . Thus the Jacobi operator on a real hypersurface M of $G_2(\mathbb{C}^{m+2})$ with respect to ξ (resp. N) is said to be a *structure Jacobi operator* (resp. *normal Jacobi operator*) and will be denoted by R_ξ (resp. \bar{R}_N).

For a commuting problem concerned with structure Jacobi operator R_ξ and structure tensor ϕ of M in $G_2(\mathbb{C}^{m+2})$, that is, $R_\xi\phi = \phi R_\xi$, Suh and Yang [14] gave a characterization of a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$. Also, concerned with commuting problem for the normal Jacobi operator \bar{R}_N , Pérez, Jeong and Suh [11] gave a characterization of a real hypersurface of Type (A) in $G_2(\mathbb{C}^{m+2})$.

On the other hand, another commuting problem $(R_\xi\phi)A = A(R_\xi\phi)$ (resp. $(\bar{R}_N\phi)A = A(\bar{R}_N\phi)$) related to the shape operator A and the restricted structure Jacobi operator $R_\xi\phi$ (resp. the restricted normal Jacobi operator $\bar{R}_N\phi$), which can be only defined in the orthogonal complement $[\xi]^\perp$ of the Reeb vector field $[\xi]$, was recently classified in [10].

Motivated by these results, let us consider the Ricci tensor S instead of the shape operator A for M in $G_2(\mathbb{C}^{m+2})$. Then as a generalization, naturally, we consider a new commuting condition for the restricted structure Jacobi operator $R_\xi\phi$ and the Ricci tensor S defined in such a way that

$$(C-1) \quad (R_\xi\phi)S = S(R_\xi\phi).$$

The geometric meaning of (C-1) can be explained in such a way that any eigenspace of R_ξ on the distribution $\mathfrak{h} = \{X \in T_x M \mid X \perp \xi\}$, $x \in M$, is invariant by the

Ricci tensor S of M in $G_2(\mathbb{C}^{m+2})$. Now we want to give a complete classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with (C-1) as follows:

Theorem 1. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $(R_\xi \phi)S = S(R_\xi \phi)$. If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ , then M is locally congruent with an open part of a tube of some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Next, we want to consider another commuting condition between the restricted normal Jacobi operator $\bar{R}_N \phi$ and the Ricci tensor S defined by

$$(C-2) \quad (\bar{R}_N \phi)S = S(\bar{R}_N \phi),$$

and give a classification of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with (C-2) as follows:

Theorem 2. *Let M be a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$ with $(\bar{R}_N \phi)S = S(\bar{R}_N \phi)$. If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ , then M is locally congruent to an open part of a tube of some radius $r \in (0, \frac{\pi}{2\sqrt{2}})$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

Actually, according to the geometric meaning of the condition (C-1)(resp. (C-2)), we also assert that any eigenspaces of the Ricci tensor S on M in $G_2(\mathbb{C}^{m+2})$ are invariant under the restricted structure Jacobi operator $R_\xi \phi$ (resp. the restricted normal Jacobi operator $\bar{R}_N \phi$). In Sections 1 and 2, we give a complete proof of Theorems 1 and 2, respectively. We refer to [1], [3], [4] and [9] for Riemannian geometric structures of $G_2(\mathbb{C}^{m+2})$, $m \geq 3$.

1. PROOF OF THEOREM 1

In this section, by using geometric quantities in [13] and [14], we give a complete proof of Theorem 1. To prove it, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with (C-1), that is,

$$(1.1) \quad (R_\xi \phi)SX = S(R_\xi \phi)X.$$

From now on, X, Y and Z always stand for any tangent vector fields on M .

Let us introduce the Ricci tensor S and structure Jacobi operator R_ξ , briefly. The curvature tensor $R(X, Y)Z$ of M in $G_2(\mathbb{C}^{m+2})$ can be derived from the curvature tensor $\bar{R}(X, Y)Z$ of $G_2(\mathbb{C}^{m+2})$. Then by contracting and using the geometric structure $JJ_\nu = J_\nu J$ ($\nu = 1, 2, 3$) related to the Kähler structure J and the quaternionic Kähler structure J_ν ($\nu = 1, 2, 3$), we can derive the Ricci tensor S given by

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \dots, e_{4m-1}\}$ denotes a basis of the tangent space $T_x M$ of M , $x \in M$, in $G_2(\mathbb{C}^{m+2})$ (see [13]).

From the definition of the Ricci tensor S and fundamental formulas in [13, section 2], we have

$$\begin{aligned}
 SX &= \sum_{i=1}^{4m-1} R(X, e_i)e_i \\
 (1.2) \quad &= (4m+7)X - 3\eta(X)\xi + hAX - A^2X \\
 &\quad + \sum_{\nu=1}^3 \{-3\eta_\nu(X)\xi_\nu + \eta_\nu(\xi)\phi_\nu\phi X - \eta(\phi_\nu X)\phi_\nu\xi - \eta(X)\eta_\nu(\xi)\xi_\nu\},
 \end{aligned}$$

where h denotes the trace of A , that is, $h = \text{Tr}A$ (see [12, (1.4)]). By inserting $Y = Z = \xi$ into the curvature tensor $R(X, Y)Z$ and using the condition of being Hopf, the structure Jacobi operator R_ξ becomes

$$\begin{aligned}
 R_\xi(X) &= R(X, \xi)\xi \\
 (1.3) \quad &= X - \eta(X)\xi - \sum_{\nu=1}^3 \left\{ \eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \right. \\
 &\quad \left. + 3g(\phi_\nu X, \xi)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X \right\} + \alpha AX - \alpha^2\eta(X)\xi
 \end{aligned}$$

(see [5, section 4]).

Using these equations (1.1), (1.2) and (1.3), we prove that the Reeb vector field ξ of M belongs to either \mathcal{Q} or \mathcal{Q}^\perp .

Lemma 1.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with (C-1). If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ , then ξ belongs to either the distribution \mathcal{Q} or the distribution \mathcal{Q}^\perp .*

Proof. In order to prove this lemma, we put

$$(1.4) \quad \xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$$

for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

In the case of $\alpha = 0$, by virtue of $Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^3 \eta_\nu(\xi)\eta_\nu(\phi Y)$ in [3, Lemma 1], we obtain easily that ξ belongs to either \mathcal{Q} or \mathcal{Q}^\perp .

Thus, we consider the next case $\alpha \neq 0$. Putting $X = \xi$ in (1.1) and using the fact $\phi\xi = 0$, it follows that

$$(1.5) \quad (R_\xi\phi)S\xi = 0.$$

From (1.2) and (1.4), we have

$$(1.6) \quad S\phi X_0 = \sigma\phi X_0,$$

$$(1.7) \quad SX_0 = (4m+7+h\alpha-\alpha^2)X_0 + \eta_1^2(\xi)X_0 - \eta(X_0)X_0,$$

$$(1.8) \quad S\xi = (4m+4+h\alpha-\alpha^2)\xi - 4\eta_1(\xi)\xi_1,$$

where $\sigma := 4m+8+h\kappa+\kappa^2$.

Multiplying ϕ to (1.8), we have

$$(1.9) \quad \phi S\xi = -4\eta(\xi_1)\phi\xi_1.$$

From $\phi\xi = 0$, we obtain $\phi_1\xi = \eta(X_0)\phi_1X_0$ and $\phi X_0 = -\eta(\xi_1)\phi_1X_0$. Because of $\eta(X_0)\eta(\xi_1) \neq 0$ and (1.9), (1.5) becomes

$$(1.10) \quad 0 = R_\xi(\phi\xi_1) = R_\xi(\phi_1X_0) = R_\xi(\phi X_0).$$

By substituting $X = \phi X_0$ into (1.3) and using (1.10), we get

$$(1.11) \quad A\phi X_0 = -\frac{4\eta^2(X_0)}{\alpha}\phi X_0.$$

Due to [5, Equation (2.10)], $A\xi_1 = \alpha\xi_1$ is derived from $\xi\alpha = 0$. This leads to

$$(1.12) \quad A\phi X_0 = \kappa\phi X_0,$$

where $\kappa = \frac{\alpha^2 + 4\eta^2 X_0}{\alpha}$ (see [5, section 4]).

Combining (1.11) and (1.12), we obtain

$$\{\alpha^2 + 8\eta^2(X_0)\}\phi X_0 = 0.$$

This means $\phi X_0 = 0$ which gives rise to a contradiction. Thus this lemma is proved. \square

Now, we shall divide our consideration into two cases that ξ belongs to either \mathcal{Q}^\perp or \mathcal{Q} , respectively. Next, we further study the case $\xi \in \mathcal{Q}^\perp$. We may put $\xi = \xi_1 \in \mathcal{Q}^\perp$ for our convenience sake.

Lemma 1.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field ξ belongs to \mathcal{Q}^\perp , then the Ricci tensor S commutes with the shape operator A , that is, $SA = AS$.*

Proof. Differentiating $\xi = \xi_1$ along any direction $X \in TM$ and using [8, section 2, (2.2) and (2.3)], it gives us

$$(1.13) \quad \phi AX = \nabla_X \xi = \nabla_X \xi_1 = q_3(X)\xi_2 - q_2(X)\xi_3 + \phi_1 AX.$$

Taking the inner product with ξ_2 and ξ_3 in (1.13), respectively gives $q_3(X) = 2\eta_3(AX)$ and $q_2(X) = 2\eta_2(AX)$. Then (1.13) can be revised:

$$(1.14) \quad \phi AX = 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX.$$

From this, by applying the inner product with any tangent vector Y , we have

$$g(\phi AX, Y) = 2\eta_3(AX)g(\xi_2, Y) - 2\eta_2(AX)g(\xi_3, Y) + g(\phi_1 AX, Y).$$

Then, by using the symmetric (resp. skew-symmetric) property of the shape operator A (resp. the structure tensor field ϕ), we have

$$-g(X, A\phi Y) = 2g(X, A\xi_3)g(\xi_2, Y) - 2g(X, A\xi_2)g(\xi_3, Y) - g(Y, A\phi_1 X)$$

for any tangent vector fields X and Y on M . Then it can be rewritten as below:

$$(1.15) \quad A\phi X = 2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X.$$

Note. Hereafter, the process used from (1.14) to (1.15) will be expressed as “taking a symmetric part of (1.14)”.

Bearing in mind that $\xi = \xi_1 \in \mathcal{Q}^\perp$, (1.2) is simplified:

$$(1.16) \quad \begin{aligned} SX &= (4m+7)X - 7\eta(X)\xi - 2\eta_2(X)\xi_2 \\ &\quad - 2\eta_3(X)\xi_3 + \phi_1\phi X + hAX - A^2X. \end{aligned}$$

Multiplying ϕ_1 to (1.16) and using basic formulas in [7, Section 2], we have

$$(1.17) \quad \phi_1\phi AX = 2\eta_3(AX)\xi_3 + 2\eta_2(AX)\xi_2 - AX + \alpha\eta(X)\xi.$$

By replacing X as AX into (1.16) and using (1.17), we obtain

$$(1.18) \quad SAX = (4m+6)AX - 6\alpha\eta(X)\xi + hA^2X - A^3X$$

and taking a symmetric part of (1.18) again, we get

$$(1.19) \quad ASX = (4m + 6)AX - 6\alpha\eta(X)\xi + hA^2X - A^3X.$$

Comparing (1.18) and (1.19), we conclude that

$$SAX = ASX$$

for any tangent X . □

By the way, we have equations (1.13) and (1.15) for the Ricci tensor likewise related to the shape operator. We may consider similar ones about the Ricci tensor as below:

Lemma 1.3. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector field ξ belongs to \mathcal{Q}^\perp , we have the following formulas*

- (i) $\phi SX = 2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \phi_1SX + \text{Rem}(X)$ and
- (ii) $S\phi X = 2\eta_3(X)S\xi_2 - 2\eta_2(X)S\xi_3 + S\phi_1X + \text{Rem}(X)$,

where the remainder term $\text{Rem}(X)$ is denoted by $\text{Rem}(X) = 4(m+2)\{2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + \phi X - \phi_1X\}$.

Proof. Multiplying ϕ to (1.16), we get the equivalent equation of the Left side of (i) as follows:

$$(1.20) \quad \phi SX = (4m + 7)\phi X - \phi_1X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + h\phi AX - \phi A^2X.$$

Using (1.14), and (1.15), the right side of (i) is can be replaced by

$$(1.21) \quad \begin{aligned} & 2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \phi_1SX + \text{Rem}(X) \\ &= 2\eta_3((4m+7)X - 2\eta_3(X)\xi_3 + \phi_1\phi X + hAX - A^2X)\xi_2 \\ & \quad - 2\eta_2((4m+7)X - 2\eta_2(X)\xi_2 + \phi_1\phi X + hAX - A^2X)\xi_3 \\ & \quad + (4m+7)\phi_1X - 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3 - \phi X + h\phi_1AX - \phi_1A^2X \\ & \quad + \text{Rem}(X) \\ &= (4m+7)\phi X - \phi_1X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + h\phi AX - \phi A^2X. \end{aligned}$$

Combining (1.20) and (1.21), we get the equation (i). In addition, (ii) can be obtained by taking a symmetric part of (i). □

By virtue of Lemmas 1.2 and 1.3, we assert the following:

Lemma 1.4. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with (C-1). If $\xi \in \mathcal{Q}^\perp$, we have $A(\phi S - S\phi) = (\phi S - S\phi)A$.*

Proof. By (i) (resp. (ii)) in Lemma 1.3, we have the left side of (1.1) (the right side of (1.1)) as follows:

$$(1.22) \quad \begin{cases} R_\xi \phi SX = 2\phi SX + \alpha A\phi SX + \text{Rem}(X), \\ SR_\xi \phi X = 2S\phi X + \alpha SA\phi X + \text{Rem}(X). \end{cases}$$

Combining equations in (1.22), we have

$$(1.23) \quad R_\xi \phi SX - SR_\xi \phi X = 2\phi SX + \alpha A\phi SX - 2S\phi X - \alpha SA\phi X = 0.$$

Case 1: $\alpha = 0$. Equation (1.23) becomes $S\phi X = \phi SX$. By virtue of [13, Theorem], we conclude that if M is a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with (1.1), then M satisfies the condition of Type (A).

Thus, we may assume the following case.

Case 2: $\alpha \neq 0$.

Using Lemma 1.2, (1.23) becomes

$$(1.24) \quad 2(\phi S - S\phi) + \alpha(A\phi S - AS\phi) = 0.$$

Taking a symmetric part of (1.24), we have

$$(1.25) \quad 2(\phi S - S\phi) - \alpha(S\phi A - \phi SA) = 0.$$

Combining (1.24) and (1.25), we know

$$(1.26) \quad A(\phi S - S\phi) = (\phi S - S\phi)A.$$

□

Lemma 1.5. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. If M satisfies $A(\phi S - S\phi) = (\phi S - S\phi)A$ and $\xi \in \mathcal{Q}^\perp$, then we have $S\phi = \phi S$.*

Proof. Since the shape operator A and the tensor $\phi S - S\phi$ are both symmetric operators and commute with each other, they are diagonalizable. So there exists a common basis $\{E_1, E_2, \dots, E_{4m-1}\}$ such that the shape operator A and the tensor $\phi S - S\phi$ both can be diagonalizable. In other words, $AE_i = \lambda_i E_i$ and $(\phi S - S\phi)E_i = \beta_i E_i$, where λ_i and β_i are scalars for all $i \in 1, 2, \dots, 4m-1$.

Here replacing X by ϕX in (1.16) (resp. multiplying ϕ to (1.16)), we have

$$(1.27) \quad \begin{cases} S\phi X = (4m+7)\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + hA\phi X - A^2\phi X, \\ \phi SX = (4m+7)\phi X - \phi_1 X + 2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + h\phi AX - \phi A^2 X. \end{cases}$$

Combining equations in (1.27), we get

$$(1.28) \quad S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2 X.$$

Putting $X = E_i$ into (1.28) and using $AE_i = \lambda_i E_i$, we obtain

$$(1.29) \quad (S\phi - \phi S)E_i = hA\phi E_i - A^2\phi E_i - h\lambda_i\phi E_i + \phi\lambda_i^2 E_i.$$

Taking the inner product with E_i into (1.29), we have

$$\beta_i g(E_i, E_i) = h\lambda_i g(\phi E_i, E_i) - \lambda_i^2 g(\phi E_i, E_i) = 0.$$

Since $g(E_i, E_i) \neq 0$, $\beta_i = 0$ for all $i \in 1, 2, \dots, 4m-1$. This is equivalent to $(S\phi - \phi S)E_i = 0$ for all $i \in 1, 2, \dots, 4m-1$. It follows that $S\phi X = \phi SX$ for any tangent vector field X on M . □

Summing up Lemmas 1.2, 1.3, 1.4, 1.5 and [13, Theorem], we conclude that if M is a Hopf hypersurface in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ satisfying (C-1), then M satisfies the condition of Type (A).

Hereafter, let us check whether the Ricci tensor of a model space of Type (A) satisfies the commuting condition (C-1).

From (1.2) and [3, Proposition 3], we obtain the following equations:

$$SX = \begin{cases} (4m + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\ (4m + 6 + h\beta - \beta^2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\ (4m + 6 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\ (4m + 8)X & \text{if } X \in T_\mu \end{cases}$$

$$R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ (\alpha\beta + 2)\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\ (\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\ 0 & \text{if } X \in T_\mu \end{cases}$$

$$(R_\xi\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ (\alpha\beta + 2)\phi\xi_\nu & \text{if } X = \xi_\nu \in T_\beta \\ (\alpha\lambda + 2)\phi X & \text{if } X \in T_\lambda \\ 0 & \text{if } X \in T_\mu. \end{cases}$$

Combining above three formulas, it follows that

$$(R_\xi\phi)SX - S(R_\xi\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 0 & \text{if } X = \xi_\nu \in T_\beta \\ 0 & \text{if } X \in T_\lambda \\ 0 & \text{if } X \in T_\mu. \end{cases}$$

Remark 1.6. When $\xi \in \mathcal{Q}^\perp$, a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with (C-1) is locally congruent to of Type (A) by virtue of [13, Theorem].

When $\xi \in \mathcal{Q}$, a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with (C-1) is locally congruent to of Type (B) by virtue of [9, Main Theorem].

Now let us consider our problem for a model space of Type (B) which will be denoted by M_B . In order to do this, let us calculate $(R_\xi\phi)S = SR_\xi\phi$ related to the M_B . On $T_x M_B$, $x \in M_B$, the equations (1.2) and (1.3) are reduced to the following equations, respectively:

(1.30)

$$SX = (4m + 7)X - 3\eta(X)\xi + hAX - A^2X - \sum_{\nu=1}^3 \{3\eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi\} \quad \text{and}$$

(1.31)

$$R_\xi(X) = X - \eta(X)\xi + \alpha AX - \alpha^2\eta(X)\xi - \sum_{\nu=1}^3 \left\{ \eta_\nu(X)\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu\xi \right\}.$$

From (1.30) and (1.30) and [3, Proposition 2], we obtain the following

$$(1.32) \quad SX = \begin{cases} (4m + 4 + h\alpha - \alpha^2)\xi & \text{if } X = \xi \in T_\alpha \\ (4m + 4 + h\beta - \beta^2)\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ (4m + 8)\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (4m + 7 + h\lambda - \lambda^2)X & \text{if } X \in T_\lambda \\ (4m + 7 + h\mu - \mu^2)X & \text{if } X \in T_\mu \end{cases}$$

$$(1.33) \quad R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ \alpha\beta\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ 4\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1 + \alpha\lambda)\phi X & \text{if } X \in T_\lambda \\ (1 + \alpha\mu)\phi X & \text{if } X \in T_\mu \end{cases}$$

$$(1.34) \quad (R_\xi\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 4\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ -\alpha\beta\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1 + \alpha\mu)\phi X & \text{if } X \in T_\lambda \\ (1 + \alpha\lambda)\phi X & \text{if } X \in T_\mu. \end{cases}$$

From (1.32), (1.33) and (1.34), it follows that

$$(1.35) \quad (R_\xi\phi)SX - SR_\xi\phi X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 4(h\beta - \beta^2 - 4)\phi\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ \alpha\beta(h\beta - \beta^2 - 4)\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (1 + \alpha\mu)(\lambda - \mu)(h - \lambda - \mu)\phi X & \text{if } X \in T_\lambda \\ (1 + \alpha\lambda)(\mu - \lambda)(h - \lambda - \mu)\phi X & \text{if } X \in T_\mu. \end{cases}$$

By calculation, we have $\lambda + \mu = \beta$ on M_B . From (1.35), we see that M_B satisfies (C-1), only when $h = \beta$ and $h\beta - \beta^2 - 4 = 0$. This gives us to a contradiction.

Hence, we give a complete proof of Theorem 1.

2. PROOF OF THEOREM 2

For a commuting problem in quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator $\bar{R}(X, N)N \in T_x M$, $x \in M$ for real hypersurfaces M in quaternionic projective space $\mathbb{Q}P^m$ or in quaternionic hyperbolic space $\mathbb{Q}H^m$, where \bar{R} denotes the curvature tensor of $\mathbb{Q}P^m$ or of $\mathbb{Q}H^m$. He [2] has also shown that the curvature adaptedness, when the normal Jacobi operator commutes the shape operator A , is equivalent to the fact that the distributions \mathcal{Q} and $\mathcal{Q}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator A of M , where $T_x M = \mathcal{Q} \oplus \mathcal{Q}^\perp$, $x \in M$. In this section, by using the notion of normal Jacobi operator $\bar{R}(X, N)N \in T_x M$, $x \in M$ for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ and geometric quantities in [11] and [13], we give a complete proof of Theorem 2.

From now on, let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with

$$(2.1) \quad (\bar{R}_N\phi)SX = S(\bar{R}_N\phi)X$$

for any tangent vector field X on M . The normal Jacobi operator \bar{R}_N of M is defined by $\bar{R}_N(X) = \bar{R}(X, N)N$ for any tangent vector $X \in T_x M$, $x \in M$. In [11, Introduction], we obtain the following equation

$$(2.2) \quad \begin{aligned} \bar{R}_N(X) &= X + 3\eta(X)\xi + 3 \sum_{\nu=1}^3 \eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^3 \{\eta_\nu(\xi)\phi_\nu\phi X - \eta_\nu(\xi)\eta(X)\xi_\nu - \eta_\nu(\phi X)\phi_\nu\xi\}. \end{aligned}$$

Lemma 2.1. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with (C-2). If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ , then ξ belongs to either the distribution \mathcal{Q} or the distribution \mathcal{Q}^\perp .*

Proof. In order to prove this lemma, we assume (1.4) again, for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^\perp$ and $\eta(X_0)\eta(\xi_1) \neq 0$.

On the other hand, from (2.2) and (1.4), we have

$$(2.3) \quad \bar{R}_N X_0 = 4\eta^2(X_0)X_0 + 4\eta_1(\xi)\eta(X_0)\xi_1 \quad \text{and}$$

$$(2.4) \quad \bar{R}_N \xi = 4\xi + 4\eta_1(\xi)\xi_1.$$

Using (1.7), (1.8), (2.3), (2.4) and inserting $X = \phi X_0$ into (2.1), we have the following equations:

$$(2.5) \quad \begin{aligned} \text{the left side of (2.1)} &= (\bar{R}_N \phi)S\phi X_0 = \sigma \bar{R}_N \phi^2 X_0 \\ &= -\sigma \bar{R}_N X_0 + \sigma \eta(X_0) \bar{R}_N \xi \\ &= -\sigma \{4\eta^2(X_0)X_0 + 4\eta_1(\xi)\eta(X_0)\xi_1\} \\ &\quad + \sigma \{4\eta(X_0)\xi + 4\eta(X_0)\eta_1(\xi)\xi_1\} \\ &= 4\sigma \eta(X_0)\eta_1(\xi)\xi_1 \end{aligned}$$

$$(2.6) \quad \begin{aligned} \text{the right side of (2.1)} &= S\bar{R}_N(\phi^2 X_0) = -S\bar{R}_N X_0 + \eta(X_0)S\bar{R}_N \xi \\ &= -4\eta^2(X_0)SX_0 - 4\eta(\xi)\eta(X_0)S\xi_1 \\ &\quad + 4\eta(X_0)S\xi + 4\eta(X_0)\eta(\xi_1)S\xi_1 \\ &= -4\eta^2(X_0)\{(4m+7+\alpha h-\alpha^2)X_0 - 3\eta(X_0)\xi \\ &\quad + \eta_1^2(\xi)X_0 - \eta(X_0)\eta_1(\xi)\xi_1\} \\ &\quad + 4\eta(X_0)\{(4m+4+\alpha h-\alpha^2)\xi - 4\eta_1(\xi)\xi_1\}, \end{aligned}$$

where $\sigma := 4m+8+h\kappa+\kappa^2$. Recalling that $\eta(X_0) \neq 0$ and combining (2.5) and (2.6), we have

$$\begin{aligned} 4\sigma \eta(X_0)\eta_1(\xi)\xi_1 &= -4\eta^2(X_0)\{(4m+7+\alpha h-\alpha^2)X_0 - 3\eta(X_0)\xi \\ &\quad + \eta_1^2(\xi)X_0 - \eta(X_0)\eta_1(\xi)\xi_1\} \\ &\quad + 4\eta(X_0)\{(4m+4+\alpha h-\alpha^2)\xi - 4\eta_1(\xi)\xi_1\}. \end{aligned}$$

Taking the inner product of above equation with X_0 , we get

$$\begin{aligned} 0 &= -4\eta(X_0)\{(4m+7+\alpha h-\alpha^2) - 3\eta^2(X_0) + \eta_1^2(\xi)\} \\ &\quad + 4\{(4m+4+\alpha h-\alpha^2)\eta(X_0)\} \\ &= -4\eta(X_0)\{3 - 3\eta^2(\xi) + \eta_1^2(\xi)\} \\ &= -16\eta(X_0)\eta_1^2(\xi). \end{aligned}$$

This gives a contradiction. Thus, we give a complete proof of this lemma. \square

Now this case implies that ξ belongs to the distribution \mathcal{Q}^\perp .

Lemma 2.2. *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with (2.1). If $\xi \in \mathcal{Q}^\perp$, we have $S\phi = \phi S$.*

Proof. Putting $\xi = \xi_1 \in \mathcal{Q}^\perp$ for our convenience sake, (2.2) becomes

$$\bar{R}_N(X) = X + 7\eta(X)\xi + 2\eta_2(X)\xi_2 + 2\eta_3(X)\xi_3 - \phi_1\phi X.$$

Because of (i) and (ii) in lemma 1.3, we have the following equations:

$$(2.7) \quad \begin{cases} \bar{R}_N\phi SX = 2\phi SX - \text{Rem}(X), \\ S\bar{R}_N\phi X = 2S\phi X - \text{Rem}(X), \end{cases}$$

where $\text{Rem}(X) = 4(m+2)\{2\eta_2(X)\xi_3 - 2\eta_3(X)\xi_2 + \phi X - \phi_1 X\}$.

Combining equations in (2.7), we conclude that (2.1) is equivalent to $S\phi X = \phi SX$. \square

In the case of $\xi \in \mathcal{Q}^\perp$, by using (i) and (ii) in Lemma 1.3, and Lemma 2.2, we can be easily seen that the commuting condition $S\phi = \phi S$ is equivalent to $(\bar{R}_N\phi)S = S(\bar{R}_N\phi)$.

Therefore, by Lemma 2.2 and [13, Theorem], we can assert that:

Remark 2.3. Real hypersurfaces of Type (A) in $G_2(\mathbb{C}^{m+2})$ satisfies the condition (C-2).

When $\xi \in \mathcal{Q}$, a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ with (C-2) is locally congruent to of Type (B) by virtue of [9, Main Theorem].

Let us consider our problem for a model space of Type (B) which will be denoted by M_B . In order to do this, let us calculate $(\bar{R}_N\phi)S = S(\bar{R}_N\phi)$ of M_B . From [3, Proposition 2], we obtain

$$(2.8) \quad \bar{R}_N(X) = \begin{cases} 4\xi & \text{if } X = \xi \in T_\alpha \\ 4\xi_\ell & \text{if } X = \xi_\ell \in T_\beta \\ 0 & \text{if } X = \phi\xi_\ell \in T_\gamma \\ X & \text{if } X \in T_\lambda \\ X & \text{if } X \in T_\mu, \end{cases}$$

$$(2.9) \quad (\bar{R}_N\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 0 & \text{if } X = \xi_\ell \in T_\beta \\ -4\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ \phi X & \text{if } X \in T_\lambda \\ \phi X & \text{if } X \in T_\mu. \end{cases}$$

From (2.8) and (2.9), it follows that

$$(\bar{R}_N\phi)SX - S(\bar{R}_N\phi)X = \begin{cases} 0 & \text{if } X = \xi \in T_\alpha \\ 0 & \text{if } X = \xi_\ell \in T_\beta \\ 4(h\beta - \beta^2 - 4)\xi_\ell & \text{if } X = \phi\xi_\ell \in T_\gamma \\ (\lambda - \mu)(h - \lambda - \mu)\phi X & \text{if } X \in T_\lambda \\ (\mu - \lambda)(h - \lambda - \mu)\phi X & \text{if } X \in T_\mu. \end{cases}$$

We see that M_B satisfies (C-2), only when $h = \beta$ and $h\beta - \beta^2 - 4 = 0$. This gives us to a contradiction.

Thus, we can give a complete proof of Theorem 2 in the introduction.

REFERENCES

- [1] D. V. Alekseevskii, *Compact quaternion spaces*, Func. Anal. Appl. **2** (1966), 106–114.
- [2] J. Berndt, *Real hypersurfaces in quaternionic space forms*, J. Reine Angew. Math. **419** (1991), 9–26.
- [3] J. Berndt and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatsh. Math. **127** (1999), 1–14.
- [4] J. Berndt and Y.J. Suh, *Isometric flows on real hypersurfaces in complex two-plane Grassmannians*, Monatsh. Math. **137** (2002), 87–98.
- [5] I. Jeong, Machado, Carlos J. D. Pérez and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with \mathcal{D}^\perp -parallel structure Jacobi operator*, Inter. J. Math. **22** (2011), no. 5, 655–673.
- [6] I. Jeong, Y. J. Suh, and M. M. Tripathi, *Real hypersurfaces of Type A in complex two-plane Grassmannians related to the normal Jacobi operator*, Bull. Korean Math. Soc. **49** (2012), no. 2, 423–434.
- [7] H. Lee, Y.S. Choi, and C. Woo, *Hopf hypersurfaces in complex two-plane Grassmannians with Reeb parallel shape operator*, Bull. Malaysian Math. Soc. (2014) (in press).
- [8] H. Lee, S. Kim, and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with certain commuting condition*, Czechoslovak Math. J. **62** (2012), no. 3, 849–861.
- [9] H. Lee and Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector*, Bull. Korean Math. Soc. **47** (2010), no. 3, 551–561.
- [10] H. Lee, Y.J. Suh and C. Woo, *Real hypersurfaces with commuting Jacobi operators in complex two-plane Grassmannians*, Houston J. Math. (2014) (in press).
- [11] J. D. Pérez, I. Jeong and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with commuting normal Jacobi operator*, Acta Math. Hungar. **117** (2007), 201–217.
- [12] J.D. Pérez and Y.J. Suh, *The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians*, J. Korean Math. Soc. **44** (2007), 211–235.
- [13] Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with commuting Ricci tensor*, J. Geom. Phys. **60** (2010), 1792–1805.
- [14] Y. J. Suh and H. Y. Yang, *Real hypersurfaces in complex two-plane Grassmannians with commuting structure Jacobi operator*, Bull. Korean Math. Soc. **45** (2008), no. 3, 495–507.

EUNMI PAK, YOUNG JIN SUH AND CHANGHWA WOO

DEPARTMENT OF MATHEMATICS,

KYUNGPOOK NATIONAL UNIVERSITY,

DAEGU 702-701, REPUBLIC OF KOREA

E-mail address: empak@knu.ac.kr

E-mail address: yjsuh@knu.ac.kr

E-mail address: legalgwch@knu.ac.kr